

From baroclinic instability to developed turbulence

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Résumé :

Un écoulement de base combinant cisaillement, rotation et stratification verticale en densité, avec les paramètres respectifs S (taux de cisaillement), f (paramètre de Coriolis) et N (fréquence de Brunt-Waisala), satisfait l'équation d'Helmholtz à condition d'ajouter une composante horizontale au gradient moyen de densité. Cette situation reproduit les conditions d'apparition de l'instabilité barocline, mais en permettant une décomposition du champ fluctuant en modes de Fourier advectés par le cisaillement moyen, en accord avec l'homogénéité statistique restreinte aux fluctuations. Il est alors possible de calculer la réponse linéaire, et donc la dynamique de "distorsion rapide" en fonction du nombre de Richardson N^2/S^2 et du rapport barocline $\epsilon = Sf/N$, puis de comparer ces résultats à une simulation numérique directe de type pseudo-spectral en boîte cisaillée, afin d'ajouter la dynamique non-linéaire et l'étude de structures instantanées. Une partie de la réponse linéaire paramétrique est fournie par un modèle sans pression, qui donne analytiquement les évolutions des tensions de Reynolds dans le cas instable $R_i < 1$, mais ne peut décrire la stabilité linéaire au voisinage de $R_i = 1$. Les résultats de simulation directe prolongent ces analyses; ils peuvent être aussi considérés comme une contribution à la turbulence développée en présence de stratification et de rotation, soumise à un forçage physiquement pertinent dans l'atmosphère et l'océan: l'instabilité barocline. Enfin, la stratégie de notre équipe est mise en perspective à cette occasion avec les communautés utilisant la même formulation spectrale.

Abstract :

The coupled effects of mean shear, density-stratification and system rotation are investigated in the context of strong turbulence, i.e. accounting for the baroclinic instability. Although there exists a large literature in the rotating shear case and the stratified shear case, with linear approaches, Direct or Large Eddy Simulations, very few studies consider the combined three ingredients in the context of distorted homogeneous turbulence.

One first has to define an admissible flow condition for including all three effects, in order to be able to treat properly the homogeneity condition in the numerical simulations. Then, we solve numerically the complete nonlinear equations for the rotating stratified shear homogeneous turbulent flow, using a pseudo-spectral method. The most relevant parameters for these Direct Numerical Simulations (DNS) are chosen from a preliminary comprehensive parametric study that includes two simplified approaches: (a) Rapid Distortion Theory (RDT) with the related stability analysis technique; (b) a simplified "pressure-less" stability analysis.

Keywords : instabilité barocline, théorie linéaire spectrale, simulation numérique directe

1 Introduction

This study is in the line of a general approach to turbulent flows subjected to mean (velocity and/or density) gradients and/or body forces (e.g. Coriolis, buoyancy, Lorentz force in MHD). The linear theory, often (improperly) referred to as RDT (Rapid Distortion Theory) is used to explore the general tendencies, with a first insight to a comprehensive parametric study. Pseudo-spectral DNS's in deformed coordinates (e.g. illustrated by Rogallo [1]) are a natural continuation, taking into account explicitly the nonlinearities, but using the same characteristic lines (trajectories of the mean flow in physical space, related solutions of an eikonal equation in Fourier space) as in the linear 'RDT' problem. Of course, the results from the —unexpensive— linear approach, which permits a very large sweeping of the parameters' range, pave the way for the —expensive— DNS study.

In these combined linear/nonlinear approaches, two points are original in our team:

First, The mean flow which is the source of linear distortion of the fluctuating one is systematically chosen as a particular solution of basic equations (Navier-Stokes and coupled fields). Mathematically, this way is consistent with statistical homogeneity restricted to the fluctuating field, if statistics are considered, and with the related stability analysis using 'admissibility conditions' (e.g. Craik [2]). Physically, this allows to take

into account relevant mechanisms like the baroclinic torque in rotating stratified turbulence with vertical shear [3],[4], and the gyroscopic torque in precessing rotating flows with additional Coriolis force [5].

Second, when linear theory is used, the linear response is characterized by the most general deterministic information: a Green's function which links any realization of the fluctuating field at time t to its initial counterpart at time t_0 . This formulation allows us to predict the second-order statistics, as in conventional 'RDT', but also gives access to higher order statistics. In addition, it is possible to incorporate the Green's function as a building block in fully nonlinear models and theories, in which nonlinear terms, more or less related to stochastic stirring forces, are treated as a source in the right-hand-side of linearized equations [6].

It is perhaps useful to make a rapid survey of the linear approach. Not less than three communities are using it, with often different motivations and objectives. The fact that they use specific jargons or parlanges, and that they publish in different journals, masks their common base. Of course, we think that our general formalism based on a deterministic Green's function can help in reconciling the different avatars of the linear theory.

The RDT limit is obtained by dropping nonlinear (and often viscous) terms in the governing equations for velocity fluctuations, in the presence of a mean flow with space-uniform gradients. Emphasis is put on prediction of low-order statistics, with isotropic initial data, in order to characterize the first stages of the development of anisotropy.

Studies in the area of hydrodynamic stability have the same starting point as 'RDT', considering an extensional admissible base flow to which disturbances are superposed in the form of advected Fourier modes. The 'base flow' is the 'mean flow' of RDT, in agreement with statistical homogeneity of the fluctuating flow, whereas no reference to statistics appears in the stability analysis: *e.g.*, no need to say that the base flow is the mean one, that the disturbance flow is the fluctuating one, that use of Fourier modes for the second is consistent with statistical homogeneity.

A third community is concerned with applications to astrophysics. The linear response of turbulence to various effects of shear, density-stratification and rotation is used for a better modelisation of the turbulence in accretion disks, mainly (see, *e.g.* [7].) Analogies and partial balance between (self)-gravitational, centrifugal, and buoyancy forces are studied, not to mention the Lorentz force in the important context of the magneto-rotational instability. Linear operators are not systematically considered as dominant over the non-linear ones, as in conventional RDT, and the linear response (Green's function) is involved with stochastic nonlinear modelling [8]. A well-known application is the calculus of an alpha effect, which can be seen as a direct feed-back from small-scale turbulence to the mean flow. Identification of various coefficients, from a possible alpha term to an effective viscosity tensor, may result from the simple analysis of the linear response of the Reynolds stress tensor to the external 'mean distortion', in the presence of a stochastic force (*e.g.* [9]) with a very simple spectrum (isotropic homogeneous, white-noise in time.)

The application chosen here illustrates the baroclinic instability and its use as a physical forcing of rotating stratified turbulence [10]. The mean, or 'base' flow, with velocity U_i and buoyancy force B (within the Boussinesq approximation, details in [6],[4]) consists of three ingredients: a vertical shear, uniform in space, with rate $S = \partial U_1 / \partial x_3$, a vertical, stabilizing, mean density gradient $\partial B / \partial x_3 \propto N^2$, characterized by the Brunt-Waisala frequency N of gravity waves, and the vertical system vorticity 2Ω (or f the Coriolis parameter for applications to atmosphere and ocean.) Indices 1, 2, 3 denote the streamwise, vertical and spanwise directions of the mean shear flow, respectively. It can be shown that the mean shear flow, with or without stable stratification, is not a particular solution in a frame rotating around the vertical axis: a spurious vorticity component is generated with rate $S.f$ in the streamwise direction, looking at the Helmholtz equation for the mean flow in the rotating frame. Instead of balancing this spurious vorticity component by an artificial body force, as done by, *e.g.*, Yu and Girimaji [11] in the different context of oscillating shear, we choose to equilibrate it by a new horizontal component of the mean buoyancy (density or temperature) gradient or $\partial B / \partial x_1 = f.S / (N^2) \partial B / \partial x_2$. The mean flow pattern characterized by (S, f, N) is shown in Fig. 1. The mean flow with additional horizontal mean density gradient is an exact solution of Helmholtz equations, the exact balance allowed by the additional horizontal component of the density gradient corresponds to the geostrophic adjustment. Accordingly, the rate $\epsilon = fS / N^2$ characterizes the tilting of mean isopycnal surfaces from purely horizontal planes, in agreement with triggering a baroclinic instability. Important nondimensional parameters are

$$R_i = \frac{N^2}{S^2}, \quad Ro = \frac{S}{f}, \quad Bu = \frac{N^2}{f^2}, \quad \epsilon = \frac{Sf}{N^2}, \quad (1)$$

or the Richardson number, the Rossby number, The Burgers number and the baroclinic coefficient. Two of them can be chosen independently for the parametric analysis of the response of turbulence, in addition to the Reynolds number.

2 Linear approach, with and without effects of fluctuating pressure

Navier-Stokes equations with additional vertical buoyancy term and coupled buoyancy equation (within the Boussinesq approximation) are linearized around the previously mentioned mean flow in a rotating frame, which is characterized by S, N, f . Four equations are found for the fluctuating flow, with components u_1, u_2, u_3 (fluctuating velocity), p (fluctuating pressure) and b (fluctuating buoyancy). A fifth equation, which allows the system of equations to be closed, is the divergencefree constraint of the velocity field, which is recovered in

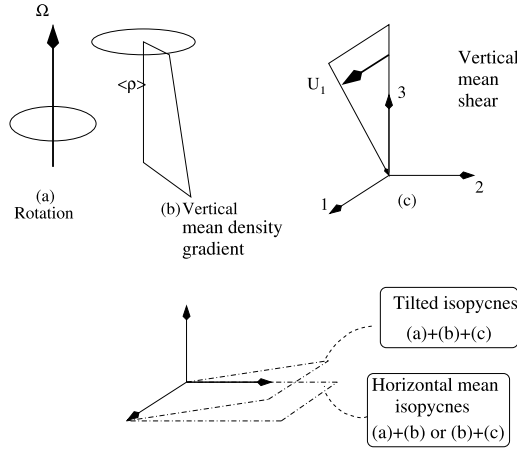


Figure 1: Sketch of the mean flow, including: (a) system vorticity; (b) vertical stable stratification; (c) mean shear. Tilting isopycnal surfaces can trigger the baroclinic instability, if (a)-(b)-(c) are simultaneously present.

the limit of low Mach number even if the density (or the buoyancy here, which replace either the density, for a liquid, or the potential temperature, for a gaz) can fluctuate around hydrostatic equilibrium.

Because of the nonlocal relationship of pressure to velocity, the general linear response would involve a functional Green's function in physical space. A more tractable solution is obtained for the Fourier components $(\hat{u}_1(\mathbf{k}, t), \hat{u}_2(\mathbf{k}, t), \hat{u}_3(\mathbf{k}, t), \hat{p}(\mathbf{k}, t), \hat{b}(\mathbf{k}, t))$, taking advantage of the algebraic form of the divergencefree constraint $\mathbf{k} \cdot \hat{\mathbf{u}} = 0$ and related removal of the pressure term. A very general solution can be written as

$$u^{(i)}(\mathbf{k}(t), t) = g_{ij}(\mathbf{k}, t, t_0) u^{(j)}(\mathbf{k}(t_0), t_0) + \int_{t_0}^t g_{ij}(\mathbf{k}, t, t') s_j(\mathbf{k}(t'), t') dt', \quad (2)$$

in terms of only three components readily derived from the basic five-component set (u_1, u_2, u_3, p, b) : two components for the velocity, $u^{(1)}$ and $u^{(2)}$, chosen in the plane normal to \mathbf{k} , and one for the buoyancy, $u^{(3)}$, scaled as a velocity component. In the previous equation, the only additional difficulty with respect to conventional Fourier analysis or Fourier synthesis, is the fact that the phase of the basic Fourier mode $\exp(i\mathbf{k}(t) \cdot \mathbf{x})$ has to be passively advected by the mean shear flow, rendering the wave vector $\mathbf{k}(t)$ time-dependent. Time-dependency is expressed by the Cauchy matrix which gives the mapping from Lagrangian to Eulerian coordinates following the mean flow trajectories, or

$$k_i = F_{ji}^{-1}(t, t_0) \underbrace{k_j(t_0)}_{K_j}, \quad \text{with} \quad x_i = F_{ij}(t, t_0) X_j. \quad (3)$$

of course, here the Cauchy matrix reduces to the simple expression $F_{ij} = \delta_{ij} + S(t - t_0)\delta_{i1}\delta_{i2}$, with corresponding eq. (3) extensively used by Townsend in RDT. More generally, the first equation in (3) gives the solution of the eikonal equation for \mathbf{k} . Incidentally, one can illustrate different terminologies in the different communities: The time-dependent Fourier mode is often called 'Kelvin mode' in the community of stability analysis, its purely shear-advected form is called a 'shear wave' in the 'astrophysical' community, whereas mean-flow-related Lagrangian coordinates \mathbf{X}, \mathbf{K} are sometime called 'Rogallo-space' in the engineering community dealing with RDT and DNS ! As a second remark about eq. (2), it is possible to take into account a source term s_i , which represents the explicit nonlinear term, a stochastic forcing, or a combination of both. Only the first term in eq. (2) gives the general linear solution of the initial-value problem.

The general form of the Green's function in eq. (2) is not discussed here for the sake of brevity. It generates the linear response and allows to discuss the stability, with exponential or algebraic growth of disturbances, or amplification or inhibition of turbulence, in terms of the external parameters (Ri, ϵ) and in terms of the orientation of the angle-dependent mode, for the initial value problem. Analysis is somewhat complicated by the time-dependency induced by \mathbf{k} , through eq. (3), in the matrix of the linear system. To what extent a much simpler analysis can predict the main trends, growth or decay ? The pressure-less analysis (PLA hereinafter) is used for this purpose. As a first remark, the success of this analysis for predicting the stability of the shear flow rotating around the spanwise direction is perhaps misleading: as mentioned by Jim Riley (private communication), the fact that the same criterion for stability can be found by a rigorous stability analysis and by a pressure-less approach, such as the displaced-particle analysis, is almost fortuitous. An explanation was

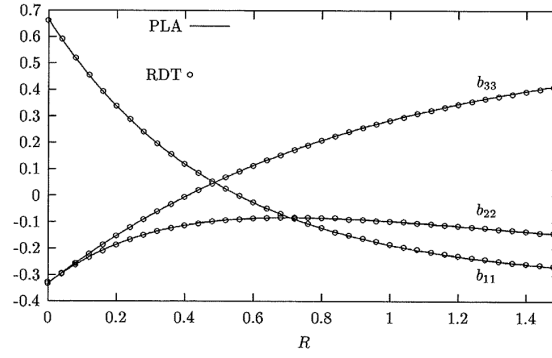


Figure 2: Deviatoric part of the Reynolds stress tensor, *i.e.* the anisotropy tensor components $b_{ij} = \overline{u_i u_j} / \overline{u_n u_n} - \delta_{ij}/3$, showing a strong growth of the vertical diagonal component b_{33} .

provided by [12] (and references therein,) who showed how the instability is governed by solenoidal modes with dominant variability in the spanwise direction, which are naturally pressureless, but are different from the set of two-component primitive variables, u_1 and u_3 here, used in the two-component displaced particle analysis. More generally, the fact that the PLA cannot ensure the solenoidal (divergencefree) property for the fluctuating velocity field is an impediment for calculating two-point statistics, but PLA can keep some relevance for calculating single-point statistics. Looking at the evolution of the Reynolds stress tensor (RST), for instance, a correct prediction by PLA means that the pressure-strain rate tensor has small impact in the RST budget, with respect to the ‘production’ tensor.

In addition, as for the case of shear rotating around the spanwise direction, the energy growth rate given by PLA is relevant in the case of dominant ‘production’ terms. Here, the PLA is consistent with instability for $Ri < 1$ since the following eigenvalues become real with a positive one that gives the growth rate:

$$\sigma = \pm \left(-(Ri + Ro^{-2}) + \sqrt{(Ri - Ro^{-2})^2 + 4Ro^{-2}} \right)^{1/2} \quad (4)$$

Single-point statistics may then be computed analytically. For instance, the anisotropy tensor components b_{ij} , *i.e.* the deviatoric part of the Reynolds stress tensor, are plotted in fig. 2. Asymptotic values can be calculated analytically in terms of Ri and Ro [4]. In this case, RDT amounts to reintroducing ‘rapid’ pressure-strain rate terms in the evolution of the RST, in agreement with the underlying linear approach in terms of solenoidal modes (first term in the r-h-s of eq. (1)). The agreement between RDT and PLA results is excellent in fig. 3. On the other hand, discrepancies appear in the stabilizing case, for $Ri = 1$ and larger values, not to mention the fact that PLA is irrelevant for predicting two-point statistics, or related spectra, even if it works for linear prediction of the RST.

3 Typical DNS results

The fluctuating fields are developed on a basis of Fourier modes in the three spatial directions, and the equations for the spectral coefficients, derived from the equations in physical space, are written in a Lagrangian framework attached to the deformable k -space, due to shear. Periodic remeshing is required to restore the skewed computational box to a cube, an operation that does not seem to induce significant energy loss. Full de-aliasing is performed when treating the nonlinear terms by direct and inverse Fourier-transforms. Finally, a second-order accurate time-stepping is applied. The curves of figures 3 show the evolution of the kinetic energy from the isotropic initial conditions, this time computed by both viscous RDT and DNS, for two parametric cases at $\epsilon = 0.2$: $Ri = 2$ and $Ri = 0.99$. As expected, the first case (figure 3-left) exhibits a steady decay of the kinetic energy, and not much difference between the linear model and the full nonlinear simulations at this limited $Re\lambda = 49$ value.

In the case at $Ri = 0.99$ (figure 3-right), the instability is present and captured by both RDT and DNS, although the growth rate is larger in the former model. This may be explained by the presence of the nonlinear energy transfer terms that start cascading a part of the energy produced by the instability at different scales than the primary instability mode. In addition, the rate of the exponential growth fitted on numerically computed RDT (fig. 3-right) is in good agreement with eq. (4).

4 Discussion of general issues in rotating stratified flows

The reader is referred to [13] for a recent study of the baroclinic instability, at a Burgers number close to 1. This suggests to refine our comparison between our ‘statistically homogeneous’ approach to more realistic flow cases subject to this instability.

On the other hand, our approach can be seen as a more physical forcing (more realistic than artificial large-scale stochastic forcings commonly used in DNS) of rotating stratified turbulence. For instance, a typical

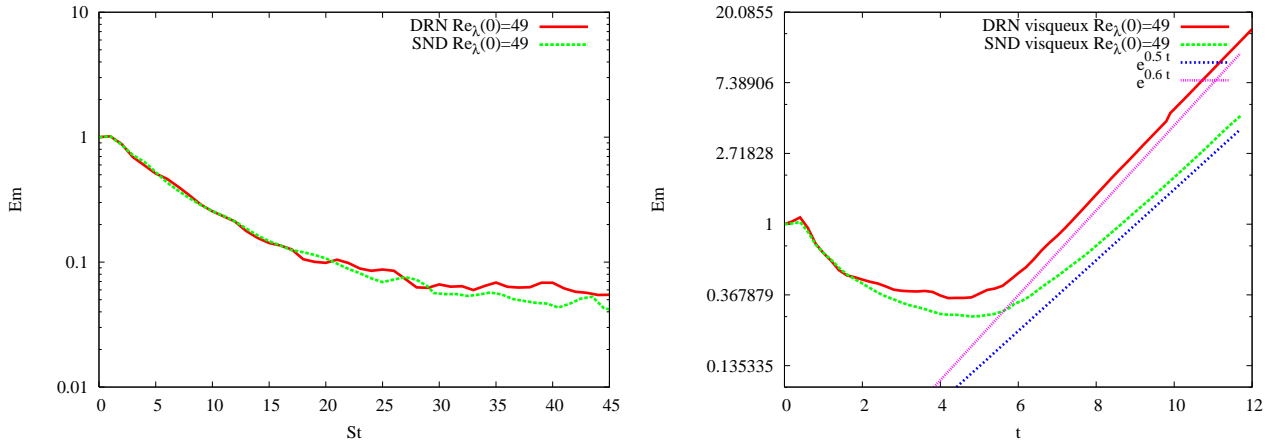


Figure 3: Time evolution of the kinetic energy computed by RDT computations (*a.k.a.* DRN in key) and 128^3 DNS at $Re_\lambda = 49$: (left) $\epsilon = 0.2$, $Ri = 2$; (right) $\epsilon = 0.2$, $Ri = 0.99$.

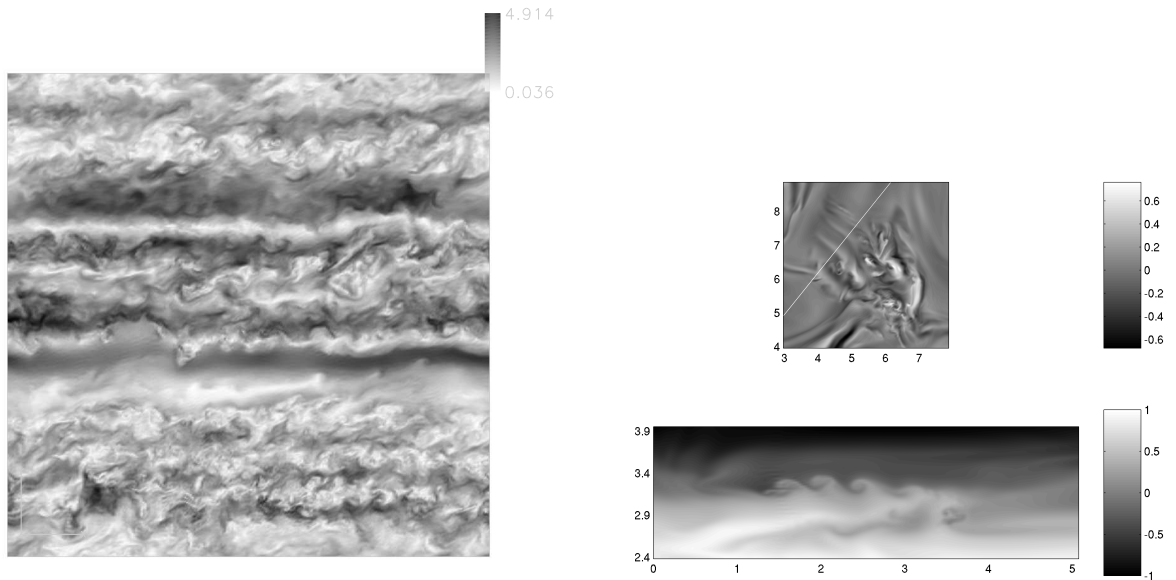


Figure 4: (left) isosurfaces of the buoyancy fluctuation in the zonal spanwise-vertical plane, from DNS at $\epsilon = 0.2$, $Ri = 0.99$, $Re_\lambda(t = 0) = 66$ [10]. (right) High Reynolds number DNS results of Riley & de Bryunkops, 2003, for stably stratified turbulence without mean shear or mean horizontal density gradient. The top panel shows part of a horizontal slice through the vertical velocity; the bottom panel shows the density on a vertical slice along the white dashed line.

structuration of the buoyancy field is shown in figure 4-left, comparing the structure of our sheared rotating stratified velocity field to that of the purely stratified case obtained by Riley and deBruynkop [14] (figure 4-right): DNS forced by baroclinic instability exhibit horizontal layering with Kelvin-Helmholtz-type structures, that resembles that in a purely vertically stratified flow at sufficiently high Reynolds number. More DNS results compared to RDT ones and contrasted with existing works will be shown in our oral presentation. A recent point is that potential vorticity remains a Lagrangian invariant even in the presence of mean shear, with exciting consequences on both linear (RDT) and nonlinear results.

Acknowledgements

The simulations have been performed thanks to cpu time allocated by IDRIS, a computing center operated by the french National Center for Scientific Research (CNRS), under project number 071851.

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